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Reproducing kernel functions of solutions to polynomial Dirac equations in the annulus of the unit ball in $\mathbb{R}^n$ and applications to boundary value problems

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ABSTRACT

Let $\mathbf{D} := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} e_i$ be the Dirac operator in $\mathbb{R}^n$ and let $P(X) = a_m X^m + \cdots + a_1 X_1 + a_0$ be a polynomial with complex coefficients. Differential equations of the form $P(\mathbf{D}) f = 0$ are called polynomial Dirac equations. In this paper we consider Hilbert spaces of Clifford algebra-valued functions that satisfy such a polynomial Dirac equation in annuli of the unit ball in $\mathbb{R}^n$. We determine an explicit formula for the Bergman kernel for solutions of complex polynomial Dirac equations of any degree m in the annulus of radii μ and 1 where $\mu \in]0, 1[$. We further give formulas for the Szegö kernel for solutions to polynomial Dirac equations of degree $m < 3$ in the annulus. This includes the Helmholtz and the Klein–Gordon equation as special cases. We further show the non-existence of the Szegö kernel for solutions to polynomial Dirac equations of degree $n \geqslant 3$ in the annulus. As an application we give an explicit representation formula for the solutions of the Helmholtz and the Klein–Gordon equation in the annulus in terms of integral operators that involve the explicit formulas of the Bergman kernel that we computed.

1. Introduction and basic notions

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and $Cl_{0n}(\mathbb{R})$ be the associated real Clifford algebra in which $e_i e_j + e_j e_i = -2\delta_{ij} e_0$, $i, j = 1, \ldots, n$, holds, δ_{ij} standing for the Kronecker symbol. Each $a \in Cl_{0n}(\mathbb{R})$ can be represented in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$, $A \subseteq \{1, \ldots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leqslant l_1 < \cdots < l_r \leqslant n$, $e_\emptyset = e_0 = 1$. The scalar part of a, $Sc(a)$, is defined as the a_0 term. The Clifford conjugate of a is defined by $\bar{a} = \sum_A a_A \bar{e}_A$, where $\bar{e}_A = \bar{e}_{l_r} \bar{e}_{l_{r-1}} \cdots \bar{e}_{l_1}$ and $\bar{e}_j = -e_j$ for $j = 1, \ldots, n$, $\bar{e}_0 = e_0 = 1$.

By forming the tensor product $Cl_{0n} \otimes_{\mathbb{R}} \mathbb{C}$ we obtain the complexified Clifford algebra $Cl_{0n}(\mathbb{C})$. Its elements are represented in the form $\sum_A a_A e_A$ where a_A are complex numbers of the form $a_A = a_{A1} + i a_{A2}$. The complex imaginary unit i commutes with all basis elements e_j. We denote the complex conjugate of a complex number $\lambda \in \mathbb{C}$ by $\lambda^\sharp$. For each $a \in Cl_{0n}(\mathbb{C})$ we have $(\bar{a})^\sharp = \overline{(a^\sharp)}$. On $Cl_{0n}(\mathbb{C})$ one considers a standard (pseudo)norm defined by $\|a\| = (\sum_A |a_A|^2)^{1/2}$. Here $|\cdot|$ is the usual absolute value of the complex number a_A.

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Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathbf{z} := x_1e_1 + \cdots + x_ne_n$ be a vector variable. Let $\mathbf{D_z} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} e_j$ be the Euclidean Dirac operator.

A real differentiable function $f : \Omega \to Cl_{0n}(\mathbb{R})$ that satisfies inside Ω the system $\mathbf{D_z} f = 0$ is called left monogenic with respect to $\mathbf{z}$. The associated function theory is often called Clifford analysis or hypercomplex analysis. It provides a higher dimensional generalization of classical complex analysis and many powerful tools to treat higher dimensional boundary value problems from harmonic analysis. See for instance [3,14,16]. Indeed, the Dirac operator factorizes the Euclidean Laplacian by $\mathbf{D_z^2} = -\Delta_{\mathbf{z}}$. Each real component of a monogenic function is therefore harmonic. This allows us to apply methods from harmonic analysis to study monogenic functions but also vice versa. Many powerful tools to do harmonic analysis with Clifford analysis methods arise from the study of Hilbert spaces of monogenic functions.

Of particular interest in this context are Hilbert spaces of Clifford-valued functions that satisfy the Bergman condition $\|f(\mathbf{z})\| \leqslant C(\mathbf{z})\|f\|_{L^2}$. Important examples are the spaces of square integrable monogenic functions over a domain of $\mathbb{R}^n$ or over its boundary. These are called Bergman spaces or Hardy spaces of monogenic functions, respectively. They have uniquely defined reproducing kernels. They are called the Bergman kernel and the Szegö kernel of monogenic functions, respectively. The associated Bergman and Szegö projections can be used for example in setting up explicit representation formulas for the solutions to the Navier–Stokes system. See for instance [16] and [11] among others. Early contributions to the study of the properties of these function spaces came from R. Delanghe and F. Brackx in 1976 and 1978, cf. [2,12]. Important follow-up contributions can found for instance in [1,6,7,22,23] and elsewhere.

To treat larger classes of partial differential equations, in particular of higher order, one also started to look at analogues of these function spaces in the more general framework of polynomial Dirac equations of the form $[\sum_{i=0}^{m} \alpha_i \mathbf{D_z^i}]f = 0$ where α_i are arbitrary complex numbers. In this setting, the functions take values in the complex Clifford algebra $Cl_{0n}(\mathbb{C})$. The associated Bergman space is equipped with the Clifford-valued inner product of the form $\langle f, g\rangle = \int_{\Omega} \overline{f(\mathbf{z})}^{\sharp} g(\mathbf{z}) dx_1 \cdots dx_n$. The derived norm has the form $\|f\|_{L^2} = \sqrt{Sc\langle f, f\rangle}$. Similarly, the associated Hardy space is equipped with the inner product of the form $\langle f, g\rangle = \int_{\partial\Omega} \overline{f(\mathbf{z})}^{\sharp} g(\mathbf{z})\, dS_{\mathbf{z}}$ where $dS_{\mathbf{z}}$ is the non-oriented surface measure.

Fundamental properties of the associated function spaces in this setting have been studied for example by F. Sommen and Xu Zhenyuan [25,27], by F. Brackx, F. Sommen, N. Van Acker [4] and by J. Ryan in [22]. See also [16,18] and elsewhere. Of central importance is the determination of explicit formulas for the reproducing kernel functions. This, however, is very difficult in general, because both the Bergman and the Szegö kernel depend on the domain. Moreover, the classical method computing the kernels by the Riemann mapping cannot be applied in dimensions $n \geqslant 3$, since there is no direct analogue of the Riemann mapping theorem. For $\lambda \neq 0$, we cannot apply the classical reflection principle from harmonic analysis, either.

In [4] an explicit representation formula for the Bergman kernel of the unit ball for the solutions of the special system $(\mathbf{D_z} - \lambda)f = 0$, for arbitrary $\lambda \in \mathbb{C}$, has been developed. J. Ryan showed in [22] that the space of solutions to $(\mathbf{D_z} - \lambda)f = 0$ that are square-integrable over a domain that has a piecewise C^1 or Lipschitz boundary, has always a uniquely defined Bergman kernel function. In [10] the first and the third author gave an explicit formula for the Bergman kernel of the unit ball associated to the more general system

$$(\mathbf{D_z} - \lambda_1)(\mathbf{D_z} - \lambda_2)\cdots(\mathbf{D_z} - \lambda_p)f(\mathbf{z}) = 0, \tag{1}$$

where $\lambda_1, \ldots, \lambda_p$ are mutually distinct arbitrary non-zero complex numbers. We also proved that the Bergman kernel does exist for any $p \in \mathbb{N}$ and for any arbitrary domain $\Omega \subset \mathbb{R}^n$. We also gave an explicit formula for the Szegö kernel of the unit ball for the systems $(\mathbf{D_z} - \lambda_1)f = 0$ and $(\mathbf{D_z} - \lambda_1)(\mathbf{D_z} - \lambda_2)f = 0$, where λ_1, λ_2 are again arbitrary distinct non-zero complex numbers. Furthermore, in [10] it was shown that there is no reproducing Szegö kernel for solutions to systems of type (1) in the unit ball if $p > 2$.

In this paper we extend the previously preformed study by establishing an explicit formula for the Bergman kernel of the space of square integrable functions that satisfy the system (1) in an arbitrary annulus of the unit ball with radii $r = \mu \in]0, 1[$ and $R = 1$. Our study also includes the cases where the elements λ_i are not pairwise distinct and where some of these values are zero. For the cases $p \leqslant 2$ we also give explicit formulas for the Szegö kernel of such annular domains. This encompasses the Helmholtz and the Klein–Gordon equation and the solutions of the time-harmonic Maxwell equations, which were treated in the Clifford calculus for instance in [17,18,20,26], as special cases. We also extend D. Calderbank's result from [5] in which the author gave an explicit formula for the Szegö kernel of the annulus of the unit ball for the system $\mathbf{D_z} f = 0$. Again the explicit knowledge of the Bergman kernel $B(\mathbf{z}, \mathbf{w})$ and the Szegö kernel $S(\mathbf{z}, \mathbf{w})$ allow us to evaluate explicitly the Bergman projection $[\mathbf{P}f](\mathbf{z}) = \int_{\Omega} B(\mathbf{z}, \mathbf{w}) f(\mathbf{w})\, dV_{\mathbf{w}}$ and the Szegö projection $[\mathbf{S}f](\mathbf{z}) = \int_{\partial\Omega} S(\mathbf{z}, \mathbf{w}) f(\mathbf{w})\, dS_{\mathbf{w}}$ where f are functions that belong for instance to some Sobolev spaces. We also show that there is no reproducing kernel in Hardy spaces for solutions to systems of the (1) in annular domains when $p > 2$. Finally, we show how the explicit representations of the Bergman kernel serve to represent explicitly solutions to generalized Helmholtz type equations with prescribed boundary data. This is presented in the last section of the paper.

2. The local representation theorem of eigensolutions to the Dirac equation

Basic tools for all that follows are the following local representation theorems for eigenfunctions of the Dirac operator with a non-zero complex eigenvalue, cf. [27]:

Lemma 1 *(Taylor series expansion). Let f be a $Cl_{0n}(\mathbb{C})$-valued function that satisfies in the n-dimensional open unit ball $B(0,1)$ the differential equation $(\mathbf{D_z}-\lambda)f(\mathbf{z})=0$ for a complex parameter $\lambda\in\mathbb{C}\setminus\{0\}$. Then there exists a sequence of spherical monogenics of total degree $q=0,1,2,\dots$, say $P_q(\mathbf{z})$, such that in each open ball $B(0,r)$ with $0<r<1$,*

$$f(\mathbf{z})=\sum_{q=0}^{+\infty}\|\mathbf{z}\|^{1-q-n/2}\left(J_{q+n/2-1}\big(\lambda\|\mathbf{z}\|\big)-\frac{\mathbf{z}}{\|\mathbf{z}\|}J_{q+n/2}\big(\lambda\|\mathbf{z}\|\big)\right)P_q(\mathbf{z}).$$

Here, J denote the usual Bessel functions of the first kind. See [15] for details.

The spherical monogenics P_q appearing in this representation are homogeneous monogenic polynomials of total degree q. They have the form

$$P_q(\mathbf{x})=\sum_{q_2+\dots+q_n=q}V_{q_2,\dots,q_n}(\mathbf{x})a_{q_2,\dots,q_n}$$

where $q_{q_2,\dots,q_n}$ are Clifford numbers and where $V_{q_2,\dots,q_n}$ stand for the Fueter polynomials. In the vector formalism the Fueter polynomials have the representation

$$V_{q_2,\dots,q_n}(\mathbf{x}):=\frac{1}{|\mathbf{q}|!}\sum\big(x_{\sigma(1)}+x_1e_1e_{\sigma(1)}\big)\dots\big(x_{\sigma(|\mathbf{q}|)}+x_1e_1e_{\sigma(|\mathbf{q}|)}\big)$$

where $|\mathbf{q}|:=q_2+\dots+q_n$ and $\sigma(i)\in\{2,\dots,n\}$. The summation is extended over all distinguishable permutations of the expressions $(x_{\sigma(i)}+x_1e_1e_{\sigma(i)})$ without repetitions.

Let us now consider functions that are eigensolutions to the Dirac equation in an annular domain of the form $B(0,\mu,1):=\{\mathbf{z}\in\mathbb{R}^n\mid\mu<\|\mathbf{z}\|<1\}$ where μ is an arbitrary real satisfying $0<\mu<1$. In annular domains, the analogue of the local representation in Lemma 1 is the following Laurent expansion representation, cf. [27]:

Lemma 2 *(Laurent series expansion). Let $0<\mu<1$. Let f be a $Cl_{0n}(\mathbb{C})$-valued function that satisfies in the n-dimensional annulus $B(0,\mu,1)$ the differential equation $(\mathbf{D_z}-\lambda)f(\mathbf{z})=0$ for a complex parameter $\lambda\in\mathbb{C}\setminus\{0\}$. Then there exist two sequences of spherical monogenics of total degree $q=0,1,2,\dots$, say $P_q(\mathbf{z})$ and $P'_q(\mathbf{z})$, such that in each annulus $B(0,r_1,r_2)$ with $0<\mu<r_1<r_2<1$:*

$$\begin{aligned}f(\mathbf{z})={}&\sum_{q=0}^{+\infty}\|\mathbf{z}\|^{1-q-n/2}\left(J_{q+n/2-1}\big(\lambda\|\mathbf{z}\|\big)-\frac{\mathbf{z}}{\|\mathbf{z}\|}J_{q+n/2}\big(\lambda\|\mathbf{z}\|\big)\right)P_q(\mathbf{z})\\&+\sum_{q'=0}^{+\infty}\|\mathbf{z}\|^{1-q'-n/2}\left(Y_{q'+n/2-1}\big(\lambda\|\mathbf{z}\|\big)-\frac{\mathbf{z}}{\|\mathbf{z}\|}Y_{q'+n/2}\big(\lambda\|\mathbf{z}\|\big)\right)P'_{q'}(\mathbf{z}).\end{aligned}$$

Here, Y denote the usual Bessel functions of the second kind.

For the proof of Lemmas 1 and 2, we refer the reader for instance to [27]. For the sake of readability, we introduce the notation $S_q(\mathbf{z},\mathbf{w})$ for the Szegö kernel for $\mathbf{D_z}$-monogenic homogeneous polynomials of total degree q in the n-dimensional unit ball $B(0,1)$, which equals

$$S_q(\mathbf{z},\mathbf{w})=\frac{(-1)^q}{A_n}\sum_{m=0}^{q}\binom{n/2-2+m}{m}\binom{n/2-1+(q-m)}{q-m}(\mathbf{zw})^m(\mathbf{wz})^{q-m}$$

where $A_n=\frac{2\pi^{n/2}}{\Gamma(n/2)}$ denotes the 'surface area' of the unit ball in $\mathbb{R}^n$.

3. The Bergman and Szegö kernel of the annulus for polynomial Dirac equations

Theorem 1. *Let $\lambda\in\mathbb{C}\setminus\{0\}$ and $\mu\in\,]0,1[$. Define the expressions f and g by*

$$f_{q,\lambda}(\mathbf{z}):=\|\mathbf{z}\|^{1-q-\frac{n}{2}}\left(J_{q+\frac{n}{2}-1}\big(\lambda\|\mathbf{z}\|\big)-\frac{\mathbf{z}}{\|\mathbf{z}\|}J_{q+\frac{n}{2}}\big(\lambda\|\mathbf{z}\|\big)\right),$$

and

$$g_{q,\lambda}(\mathbf{z}):=\|\mathbf{z}\|^{1-q-\frac{n}{2}}\left(Y_{q+\frac{n}{2}-1}\big(\lambda\|\mathbf{z}\|\big)-\frac{\mathbf{z}}{\|\mathbf{z}\|}Y_{q+\frac{n}{2}}\big(\lambda\|\mathbf{z}\|\big)\right),$$

where J denotes the usual Bessel function of first kind and Y the Bessel function of second kind. The Bergman kernel of the annulus $B(0,\mu,1)$ associated to the equation $(\mathbf{D_z}-\lambda)f=0$ has the representation

$$B_{\mu,\lambda}(\mathbf{z},\mathbf{w})=\sum_{q=0}^{\infty}\left(\big(f_{q,\lambda}(\mathbf{z}),g_{q,\lambda}(\mathbf{z})\big)S_q(\mathbf{z},\mathbf{w})\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}\begin{pmatrix}\overline{f}_{q,\lambda^\sharp}(\mathbf{w})\\\overline{g}_{q,\lambda^\sharp}(\mathbf{w})\end{pmatrix}\right).$$

Here, the matrix entries have the form

$$a = \int_{\mu}^{1} r^{2q+n-1} Sc\{\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q,\lambda}(r\omega)\}\,dr,$$

$$b = \int_{\mu}^{1} r^{2q+n-1} Sc\{\overline{f}_{q,\lambda^\sharp}(r\omega) g_{q,\lambda}(r\omega)\}\,dr,$$

$$c = \int_{\mu}^{1} r^{2q+n-1} Sc\{\overline{g}_{q,\lambda^\sharp}(r\omega) f_{q,\lambda}(r\omega)\}\,dr,$$

$$d = \int_{\mu}^{1} r^{2q+n-1} Sc\{\overline{g}_{q,\lambda^\sharp}(r\omega) g_{q,\lambda}(r\omega)\}\,dr,$$

where we put $r := \|\mathbf{w}\|$ and $\omega := \frac{\mathbf{w}}{r}$.

Remark. The elements a, b, c, d actually are constants. They do not depend on ω. This a consequence of considering only the scalar part (see also calculations below).

Proof of Theorem 1. Define $\theta := \det\binom{a\ b}{c\ d} = ad - bc$. We have

$$\langle B_{\mu,\lambda}(\mathbf{z},\mathbf{w}), f_{q',\lambda}(\mathbf{w}) S_{q'}(\mathbf{w},\mathbf{v})\rangle = \frac{1}{\theta} \int_{\mathbf{w}\in B(0,\mu,1)} \big(S_{q,}(\mathbf{z},\mathbf{w})\big(f_{q,\lambda}(\mathbf{z}) d \overline{f}_{q,\lambda^\sharp}(\mathbf{w}) f_{q',\lambda}(\mathbf{w}) + g_{q,\lambda}(\mathbf{z})(-c)\overline{f}_{q,\lambda^\sharp}(\mathbf{w}) f_{q',\lambda}(\mathbf{w})$$
$$+ f_{q,\lambda}(\mathbf{z})(-b)\overline{g}_{q,\lambda^\sharp}(\mathbf{w}) f_{q',\lambda}(\mathbf{w}) + g_{q,\lambda}(\mathbf{z}) a \overline{g}_{q,\lambda^\sharp}(\mathbf{w}) f_{q',\lambda}(\mathbf{w})\big) S_{q'}(\mathbf{w},\mathbf{v})\big)\,dV_{\mathbf{w}}.$$

We first consider the expression

$$\int_{\mathbf{w}\in B(0,\mu,1)} S_{q,}(\mathbf{z},\mathbf{w}) f_{q,\lambda}(\mathbf{z}) d \overline{f}_{q,\lambda^\sharp}(\mathbf{w}) f_{q',\lambda}(\mathbf{w}) S_{q'}(\mathbf{w},\mathbf{v})\,dV_{\mathbf{w}},$$

and represent it in polar coordinates:

$$\int_{\mu}^{1} \int_{\omega\in S} S_{q,}(\mathbf{z}, r\omega) f_{q,\lambda}(\mathbf{z}) d \overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega) S_{q'}(r\omega,\mathbf{v})\,dS_\omega r^{n-1}\,dr,$$

where $S := \{\mathbf{z} \in \mathbb{R}^n \mid \|\mathbf{z}\| = 1\}$ is the unit sphere in $\mathbb{R}^n$. Then we decompose the middle part of the integrand into its scalar and vector part:

$$\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega) = Sc\{\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega)\} + h(r)\omega.$$

The term $\omega S_{q'}(r\omega,\mathbf{v})$ is an outer spherical monogenic on $\omega \in S$. It is hence orthogonal to the expression $S_q(r\omega,\mathbf{v})$. Consequently, the contribution of $h(r)\omega$ in the integral vanishes. Only the expression

$$\int_{\mu}^{1} \int_{\omega\in S} S_{q,}(\mathbf{z}, r\omega) f_{q,\lambda}(\mathbf{z})\, d\, Sc\{\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega)\} S_{q'}(r\mathbf{w},\mathbf{v})\,dS_\omega r^{n-1}\,dr \tag{2}$$

remains. We observe that

$$Sc\{\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega)\} = r^{2-q-q'-n}\big(J_{q+\frac{n}{2}-1}(r\lambda^\sharp) J_{q'+\frac{n}{2}-1}(r\lambda) + J_{q+\frac{n}{2}}(r\lambda^\sharp) J_{q'+\frac{n}{2}}(r\lambda)\big).$$

This expression is obviously independent of $\omega \in S$, as mentioned in the remark directly before the proof. Hence, the previous integral can be written as

$$\int_{\mu}^{1} Sc\{\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega)\} f_{q,\lambda}(\mathbf{z})\, d \bigg(\int_{\omega\in S} S_q(\mathbf{z}, r\omega) S_{q'}(r\omega,\mathbf{v})\,dS_\omega\bigg) r^{n-1}\,dr.$$

As a consequence of the reproduction property of $S_q(\mathbf{z},\mathbf{w})$ and of the homogeneity property $S_q(\mathbf{z},r\omega)=r^q S_q(\mathbf{z},\omega)$ which is valid for all real r, we can also write (2) in the form

$$\delta_{q,q'} d f_{q,\lambda}(\mathbf{z}) S_q(\mathbf{z},\mathbf{v}) \int_\mu^1 Sc\left\{\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega)\right\} r^{q+q'+n-1}\,dr$$

so that we obtain

$$\sum_{q=0}^\infty \delta_{q,q'} f_{q,\lambda}(\mathbf{z}) d S_{q'}(\mathbf{z},\mathbf{v}) \int_\mu^1 Sc\left\{\overline{f}_{q,\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega)\right\} r^{q+q'+n-1}\,dr = f_{q,\lambda}(\mathbf{z}) d S_{q'}(\mathbf{z},\mathbf{v}) \int_\mu^1 Sc\left\{\overline{f}_{q',\lambda^\sharp}(r\omega) f_{q',\lambda}(r\omega)\right\} r^{2q'+n-1}\,dr$$

which in turn equals

$$f_{q,\lambda}(\mathbf{z}) a S_{q'}(\mathbf{z},\mathbf{v}) \int_\mu^1 \left(J_{q'+\frac{n}{2}-1}\left(r\lambda^\sharp\right) J_{q'+\frac{n}{2}-1}(r\lambda) + J_{q'+\frac{n}{2}}\left(r\lambda^\sharp\right) J_{q'+\frac{n}{2}}(r\lambda)\right) r\,dr = a d f_{q,\lambda}(z) S_{q'}(\mathbf{z},\mathbf{v}).$$

If we apply the same calculations to the remaining three summands, and if we sum again over q, then we obtain

$$\frac{1}{\theta}(ad-ac-bc+ac) f_{q,\lambda}(\mathbf{z}) S_{q'}(\mathbf{z},\mathbf{v}) = \frac{1}{\theta}\det(\mathcal{M}) f_{q,\lambda}(\mathbf{z}) S_{q'}(\mathbf{z},\mathbf{v}) = f_{q,\lambda}(\mathbf{z}) S_{q'}(\mathbf{z},\mathbf{v}).$$

The reproduction property now follows from the fact that the functions $f_{q,\lambda} S_{q'}$ and $g_{q,\lambda} S_{q'}$ form a generating system for the Bergman space.

It remains to verify the invertibility of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $(\alpha_1,\alpha_2)^t \in \mathbb{C}^2$ be such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0.$$

Consider the function $h := f_{q,\lambda}\alpha_1 S_q(\mathbf{z},\mathbf{w}) + g_{q,\lambda}\alpha_2 S_q(\mathbf{z},\mathbf{w})$. We have

$$\begin{aligned}
\left\langle f_{q,\lambda} S_q(\mathbf{z},\mathbf{w}), h\right\rangle &= \left\langle f_{q,\lambda} S_q(\mathbf{z},\mathbf{w}), f_{q,\lambda_j}\alpha_1 S_q(\mathbf{z},\mathbf{w}) + g_{q,\lambda}\alpha_2 S_q(\mathbf{z},\mathbf{w})\right\rangle \\
&= \alpha_1\left\langle f_{q,\lambda} S_q(\mathbf{z},\mathbf{w}), f_{q,\lambda} S_q(\mathbf{z},\mathbf{w})\right\rangle + \alpha_2\left\langle f_{q,\lambda} S_q(\mathbf{z},\mathbf{w}), g_{q,\lambda} S_q(\mathbf{z},\mathbf{w})\right\rangle \\
&= a\alpha_1 S_q(\mathbf{z},\mathbf{w}) + b\alpha_2 S_q(\mathbf{z},\mathbf{w}) \\
&= (\underbrace{a\alpha_1 + b\alpha_2}_{=0}) S_q(\mathbf{z},\mathbf{w}) = 0.
\end{aligned}$$

Analogously we obtain $\langle g_{q,\lambda} S_q(\mathbf{z},\mathbf{w}), h\rangle = c\alpha_j S_q(\mathbf{z},\mathbf{w}) + d\alpha_j S_q(\mathbf{z},\mathbf{w}) = 0$, for $j=1,2$. So, we have $h \perp f_{q,\lambda}, g_{q,\lambda}$, where the orthogonality relation has to be understood in the sense of the L^2-product involving the volume integral over annulus of the unit ball. In turn, from this orthogonality relation we obtain $h \perp (f_{q,\lambda}\alpha_1 S_q(\mathbf{z},\mathbf{w}) + g_{q,\lambda}\alpha_2 S_q(\mathbf{z},\mathbf{w}))$, so $h \perp h$ and hence $h=0$. Since $S_q \neq 0$, it follows that $f_{q,\lambda}\alpha_1 + g_{q,\lambda}\alpha_2 = 0$. Since the functions $\{f_{q,\lambda}, g_{q,\lambda}\}$ are linearly independent, it necessarily follows that $\alpha_1=\alpha_2=0$. Hence, the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible. □

More generally, we can establish

Theorem 2. *Let $0<\mu<1$, $p\in\mathbb{N}$ and $\lambda_1,\ldots,\lambda_p \in \mathbb{C}\setminus\{0\}$ be mutually distinct values. Then the reproducing Bergman kernel of the annulus with radii $r=\mu$ and $R=1$ with respect to the operator $(\mathbf{D_z}-\lambda_1)\cdots(\mathbf{D_z}-\lambda_p)$ is given by*

$$B_{\mu,\lambda_1,\ldots,\lambda_p}(\mathbf{z},\mathbf{w}) = \sum_{q=0}^\infty \begin{pmatrix} f_{q,\lambda_1} \\ \vdots \\ f_{q,\lambda_p} \\ g_{q,\lambda_1} \\ \vdots \\ g_{q,\lambda_p} \end{pmatrix}^t (\mathbf{z}) S_q(\mathbf{z},\mathbf{w}) [\mathcal{M}_{\lambda_1,\ldots,\lambda_p}]^{-1} \begin{pmatrix} \overline{f}_{q,\lambda_1^\sharp} \\ \vdots \\ \overline{f}_{q,\lambda_p^\sharp} \\ \overline{g}_{q,\lambda_1^\sharp} \\ \vdots \\ \overline{g}_{q,\lambda_p^\sharp} \end{pmatrix} (\mathbf{w}).$$

Here the entries of the matrix $\mathcal{M}_{\lambda_1,\ldots,\lambda_p} := (m_{ij})_{i,j=1}^{2p}$ *are given by*

$$m_{ij} = \begin{cases} \int_\mu^1 r^{2q+n-1} Sc\{\overline{f}_{q,\lambda_i^\sharp}(r\omega) f_{q,\lambda_j}(r\omega)\}\,dr, & i,j \in \{1,\ldots,p\},\\ \int_\mu^1 r^{2q+n-1} Sc\{\overline{g}_{q,\lambda_{i-p}^\sharp}(r\omega) f_{q,\lambda_j}(r\omega)\}\,dr, & i \in \{p+1,\ldots,2p\},\ j \in \{1,\ldots,p\},\\ \int_\mu^1 r^{2q+n-1} Sc\{\overline{f}_{q,\lambda_i^\sharp}(r\omega) g_{q,\lambda_{j-p}}(r\omega)\}\,dr, & i \in \{1,\ldots,p\},\ j \in \{p+1,\ldots,2p\},\\ \int_\mu^1 r^{2q+n-1} Sc\{\overline{g}_{q,\lambda_{i-p}^\sharp}(r\omega) g_{q,\lambda_{j-p}}(r\omega)\}\,dr, & i \in \{p+1,\ldots,2p\},\ j \in \{p+1,\ldots,2p\}, \end{cases}$$

where we put again $r := \|\mathbf{w}\|$ *and* $\omega := \frac{\mathbf{w}}{r}$.

Proof. Let $[\mathcal{M}_{\lambda_1,\ldots,\lambda_p}]^{-1} := (\tilde{m}_{ij})_{i,j=1}^{2p}$. We first show the reproduction property of the expressions $f_{q,\lambda_t}(\mathbf{w}) S_q(\mathbf{w},\mathbf{v})$. To this end we represent the expression

$$\int\limits_{\mathbf{w}\in B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w}) \overline{f}_{q,\lambda_t^\sharp}(\mathbf{w}) f_{q',\lambda_{t'}}(\mathbf{w}) S_{q'}(\mathbf{w},\mathbf{v})\,dV_{\mathbf{w}}$$

in polar coordinates, i.e.,

$$\int\limits_\mu^1 \int\limits_{\omega\in S} S_q(\mathbf{z}, r\omega) \overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega) S_{q'}(r\omega,\mathbf{v})\,dS_\omega r^{n-1}\,dr.$$

Now we decompose the integrand into its scalar and vector part:

$$\overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega) = Sc\big\{\overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega)\big\} + h(r)\omega.$$

Since the term $\omega S_{q'}(r\omega,\mathbf{v})$ is an outer spherical monogenic on $\omega \in S$, it is orthogonal to the expression $S_q(r\omega,\mathbf{v})$. Consequently the contribution of $h(r)\omega$ to the integral vanishes. Thus, only the expression

$$\int\limits_\mu^1 \int\limits_{\omega\in S} Sc\big\{\overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega)\big\} S_q(\mathbf{z}, r\omega) S_{q'}(r\omega,\mathbf{v})\,dS_\omega r^{n-1}\,dr \tag{3}$$

remains. We observe that

$$Sc\big\{\overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega)\big\} = r^{2-q-q'-n}\big(J_{q+\frac{n}{2}-1}\big(r\lambda_t^\sharp\big) J_{q'+\frac{n}{2}-1}(r\lambda_{t'}) + J_{q+\frac{n}{2}}\big(r\lambda_t^\sharp\big) J_{q'+\frac{n}{2}}(r\lambda_{t'})\big).$$

This expression is clearly independent of $\omega \in S$, so that the previous integral simplifies to

$$\int\limits_\mu^1 Sc\big\{\overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega)\big\}\bigg(\int\limits_{\omega\in S} S_q(\mathbf{z}, r\omega) S_{q'}(r\omega,\mathbf{v})\,dS_\omega\bigg) r^{n-1}\,dr.$$

As a consequence of the reproduction property of the expression $S_q(\mathbf{z},\omega)$ and in view of the homogeneity $S_q(\mathbf{z}, r\omega) = r^q S_q(\mathbf{z},\omega)$ for real r, the integral (3) can in turn be written as

$$\delta_{q,q'} S_q(\mathbf{z},\mathbf{v}) \int\limits_\mu^1 Sc\big\{\overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega)\big\} r^{q+q'+n-1}\,dr.$$

Thus, we obtain

$$\sum_{q=0}^\infty \delta_{q,q'} S_q(\mathbf{z},\mathbf{v}) \int\limits_\mu^1 Sc\big\{\overline{f}_{q,\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega)\big\} r^{q+q'+n-1}\,dr = S_{q'}(\mathbf{z},\mathbf{v}) \int\limits_\mu^1 Sc\big\{\overline{f}_{q',\lambda_t^\sharp}(r\omega) f_{q',\lambda_{t'}}(r\omega)\big\} r^{2q'+n-1}\,dr$$

which in turn equals

$$S_{q'}(\mathbf{z},\mathbf{v}) \int\limits_\mu^1 \big(J_{q'+\frac{n}{2}-1} r\big(\lambda_t^\sharp\big) J_{q'+\frac{n}{2}-1}(r\lambda_{t'}) + J_{q'+\frac{n}{2}}\big(r\lambda_t^\sharp\big) J_{q'+\frac{n}{2}}(r\lambda_{t'})\big) r\,dr.$$

Analogously, we obtain that

$$\sum_{q=0}^{\infty}\int_{\mathbf{w}\in B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w})\overline{g}_{q,\lambda_t^\sharp}(\mathbf{w}) f_{q',\lambda_{t'}}(\mathbf{w}) S_{q'}(\mathbf{w},\mathbf{v})\,dV_{\mathbf{w}} = \int_{\mu}^{1}\bigl(Y_{q'+\frac{n}{2}-1}\bigl(\lambda_t^\sharp\bigr)J_{q'+\frac{n}{2}-1}(\lambda_{t'}r) + Y_{q'+\frac{n}{2}}\bigl(\lambda_t^\sharp\bigr)J_{q'+\frac{n}{2}}(\lambda_{t'}r)\bigr)r\,dr,$$

$$\sum_{q=0}^{\infty}\int_{\mathbf{w}\in B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w})\overline{f}_{q,\lambda_t^\sharp}(\mathbf{w}) g_{q',\lambda_{t'}}(\mathbf{w}) S_{q'}(\mathbf{w},\mathbf{v})\,dV_{\mathbf{w}} = \int_{\mu}^{1}\bigl(J_{q'+\frac{n}{2}-1}\bigl(\lambda_t^\sharp\bigr)Y_{q'+\frac{n}{2}-1}(\lambda_{t'}r) + J_{q'+\frac{n}{2}}\bigl(\lambda_t^\sharp\bigr)Y_{q'+\frac{n}{2}}(\lambda_{t'}r)\bigr)r\,dr,$$

$$\sum_{q=0}^{\infty}\int_{\mathbf{w}\in B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w})\overline{g}_{q,\lambda_t^\sharp}(\mathbf{w}) g_{q',\lambda_{t'}}(\mathbf{w}) S_{q'}(\mathbf{w},\mathbf{v})\,dV_{\mathbf{w}} = \int_{\mu}^{1}\bigl(Y_{q'+\frac{n}{2}-1}\bigl(\lambda_t^\sharp\bigr)Y_{q'+\frac{n}{2}-1}(\lambda_{t'}r) + Y_{q'+\frac{n}{2}}\bigl(\lambda_t^\sharp\bigr)Y_{q'+\frac{n}{2}}(\lambda_{t'}r)\bigr)r\,dr,$$

$$\int_{\mathbf{w}\in B(0,\mu,1)}\sum_{q=0}^{\infty}(f_{q,\lambda_1},\dots,f_{q,\lambda_p},g_{q,\lambda_1},\dots,g_{q,\lambda_p})(\mathbf{z})\begin{pmatrix}\tilde{m}_{11} & \dots & \tilde{m}_{1,2p}\\ \tilde{m}_{21} & \dots & \tilde{m}_{2,2p}\\ \vdots & & \vdots\\ \tilde{m}_{2p,1} & \dots & \tilde{m}_{2p,2p}\end{pmatrix} S_q(\mathbf{z},\mathbf{w})\begin{pmatrix}\overline{f_{q,\lambda_1^\sharp}}\\ \vdots\\ \overline{f_{q,\lambda_p^\sharp}}\\ \overline{g_{q,\lambda_1^\sharp}}\\ \vdots\\ \overline{g_{q,\lambda_p^\sharp}}\end{pmatrix}(\mathbf{w}) f_{q',\lambda_i}(\mathbf{w}) S_{q'}(\mathbf{w},\mathbf{v})\,dS_{\mathbf{w}}$$

$$= \sum_{q=0}^{\infty}\int_{\mathbf{w}\in B(0,\mu,1)}\begin{pmatrix}\sum_{s=1}^{p} f_{q,\lambda_s}\tilde{m}_{s,1} + \sum_{s=p+1}^{2p} g_{q,\lambda_s-p}\tilde{m}_{s,1}\\ \vdots\\ \sum_{s=1}^{p} f_{q,\lambda_s}\tilde{m}_{s,2p} + \sum_{s=p+1}^{2p} g_{q,\lambda_s-p}\tilde{m}_{s,2p}\end{pmatrix}^t(\mathbf{z})$$

$$\times\begin{pmatrix}S_q(\mathbf{z},\mathbf{w})(\overline{f_{q,\lambda_1^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\\ \vdots\\ S_q(\mathbf{z},\mathbf{w})(\overline{f_{q,\lambda_p^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\\ S_q(\mathbf{z},\mathbf{w})(\overline{g_{q,\lambda_1^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\\ \vdots\\ S_q(\mathbf{z},\mathbf{w})(\overline{g_{q,\lambda_p^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\end{pmatrix} dS_{\mathbf{w}}$$

$$= \sum_{q=0}^{\infty}\left[\sum_{s=1}^{p} f_{q,\lambda_s}(\tilde{m}_{s,1},\dots,\tilde{m}_{s,2p}) + \sum_{s=p+1}^{2p} g_{q,\lambda_{s-p}}(\tilde{m}_{s,1},\dots,\tilde{m}_{s,2p})\right](\mathbf{z})$$

$$\times\int_{\mathbf{w}\in B(0,\mu,1)}\begin{pmatrix}(S_q(\mathbf{z},\mathbf{w})(\overline{f_{q,\lambda_1^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\\ \vdots\\ S_q(\mathbf{z},\mathbf{w})(\overline{f_{q,\lambda_p^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\\ S_q(\mathbf{z},\mathbf{w})(\overline{g_{q,\lambda_1^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\\ \vdots\\ S_q(\mathbf{z},\mathbf{w})(\overline{g_{q,\lambda_p^\sharp}}f_{q',\lambda_i})(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\end{pmatrix} dS_{\mathbf{w}}$$

$$= \left[\sum_{s=1}^{p} f_{q',\lambda_s}(\tilde{m}_{s,1},\dots,\tilde{m}_{s,2p}) + \sum_{s=p+1}^{2p} g_{q',\lambda_{s-p}}(\tilde{m}_{s,1},\dots,\tilde{m}_{s,2p})\right](\mathbf{z})\begin{pmatrix}m_{1,i}\\ \vdots\\ m_{2p,i}\end{pmatrix} S_{q'}(\mathbf{z},\mathbf{v})$$

$$= f_{q',\lambda_i}(\mathbf{z})S_{q'}(\mathbf{z},\mathbf{v}).$$

In the calculation of the last step one has to keep in mind that the ith column of $\mathcal{M}$ is successively multiplied with the rows of $\mathcal{M}^{-1}$, so that all summands except of the ith one vanish. In the case where we consider g_{q,λ_i} instead of f_{q,λ_i}, we obtain the $(i+p)$th column of $\mathcal{M}$, such that precisely the term with g_{q,λ_i} is the only one that does not vanish. All the calculations involving the functions $g_{q,\lambda_i}S_q$ can be performed completely analogously. Since the set of functions $f_{q,\lambda_i}S_q$ and the set of functions $g_{q,\lambda_i}S_q$ form a generating system of the Bergman space, the reproduction property follows. However, it still remains to verify that the matrix $\mathcal{M}$ actually is invertible. Take $(\alpha_1,\dots,\alpha_{2p})^t\in\mathbb{C}^{2p}$ such that $\sum_{j=1}^{2p} m_{ij}\alpha_j = 0$

for all $i \in \{1, \dots, 2p\}$. We consider the function $h := \sum_{j=1}^{p} f_{q,\lambda_j} \alpha_j S_q(\mathbf{z}, \mathbf{w}) + \sum_{j=p+1}^{2p} g_{q,\lambda_{j-p}} \alpha_j S_q(\mathbf{z}, \mathbf{w})$. For any arbitrary $i \in \{1, \dots, p\}$ we have

$$\begin{aligned}
\langle f_{q,\lambda_i} S_q(\mathbf{z},\mathbf{w}), h\rangle &= \left\langle f_{q,\lambda_i} S_q(\mathbf{z},\mathbf{w}), \sum_{j=1}^{p} f_{q,\lambda_j}\alpha_j S_q(\mathbf{z},\mathbf{w}) + \sum_{j=p+1}^{2p} g_{q,\lambda_{j-p}}\alpha_j S_q(\mathbf{z},\mathbf{w})\right\rangle \\
&= \sum_{j=1}^{p} \alpha_j \langle f_{q,\lambda_i} S_q(\mathbf{z},\mathbf{w}), f_{q,\lambda_j} S_q(\mathbf{z},\mathbf{w})\rangle + \sum_{j=p+1}^{2p} \alpha_j \langle f_{q,\lambda_i} S_q(\mathbf{z},\mathbf{w}), g_{q,\lambda_{j-p}} S_q(\mathbf{z},\mathbf{w})\rangle \\
&= \sum_{j=1}^{p} m_{ij}\alpha_j S_q(\mathbf{z},\mathbf{w}) + \sum_{j=p+1}^{2p} m_{ij}\alpha_j S_q(\mathbf{z},\mathbf{w}) \\
&= \underbrace{\left(\sum_{j=1}^{2p} m_{ij}\alpha_j\right)}_{=0} S_q(\mathbf{z},\mathbf{w}) = 0.
\end{aligned}$$

Analogously one obtains that

$$\langle g_{q,\lambda_i} S_q(\mathbf{z},\mathbf{w}), h\rangle = \sum_{j=1}^{p} m_{i+p,j}\alpha_j S_q(\mathbf{z},\mathbf{w}) + \sum_{j=p+1}^{2p} m_{i+p,j}\alpha_j S_q(\mathbf{z},\mathbf{w}) = 0.$$

Thus, $h \perp f_{q,\lambda_i}, g_{q,\lambda_i}$ for all $i \in \{1, \dots, p\}$. Here again, the orthogonality has to be understood in the sense of the L^2 inner product involving the volume integral over the unit ball. In turn from this orthogonal relation it follows that $h \perp (\sum_{i=1}^{p} f_{q,\lambda_i}\alpha_i S_q(\mathbf{z},\mathbf{w}) + \sum_{i=p+1}^{2p} g_{q,\lambda_{i-p}}\alpha_i S_q(\mathbf{z},\mathbf{w}))$, hence $h \perp h$ and therefore $h = 0$. In view of $S_q \neq 0$ we necessarily have

$$\sum_{i=1}^{p} f_{q,\lambda_i}\alpha_i + \sum_{i=p+1}^{2p} g_{q,\lambda_{i-p}}\alpha_i = 0.$$

Since the elements λ_i are mutually distinct the set of functions $\{f_{q,\lambda_i}, g_{q,\lambda_{i-p}};\ i \in \{1, \dots, p\}\}$ is linearly independent. Therefore, $\alpha_1 = \ldots = \alpha_{2p} = 0$. Consequently, the matrix $\mathcal{M}_{\lambda_1,\dots,\lambda_p}$ is invertible. □

Remarks. As well known, in general a complex polynomial $P(\mathbf{D}) = a_m\mathbf{D}^m + \cdots + a_1\mathbf{D} + a_0$ can have multiple zeroes and some of them of course can be zero, too. Let us discuss how the formula from Theorem 2 can be adapted in these remaining cases.

- Let us first consider the case where all zeroes of the polynomial $P(\mathbf{D})$, denoted by $\lambda_1, \dots, \lambda_m$ are non-zero complex numbers. Suppose first that for instance λ_1 and λ_2 are distinct. Then the associated functions $f_{q,\lambda_1}, f_{q,\lambda_2}, g_{q,\lambda_1}, g_{q,\lambda_2}$ are clearly linearly independent. In this case the expression $\frac{f_{q,\lambda_2} - f_{q,\lambda_1}}{\lambda_2 - \lambda_1}$ is linearly independent from f_{q,λ_1}, too. The same is true for the functions g_{q,λ_1} and $\frac{g_{q,\lambda_2} - g_{q,\lambda_1}}{\lambda_2 - \lambda_1}$. Suppose now that $\lambda_2 = \lambda_1$. This case results from considering the limit $\lambda_2 \to \lambda_1$. In this case, $\frac{f_{q,\lambda_2} - f_{q,\lambda_1}}{\lambda_2 - \lambda_1}$ tends to $\frac{\partial f_{q,\lambda_1}}{\partial \lambda_1}$. Then f_{q,λ_1} and $\frac{\partial f_{q,\lambda_1}}{\partial \lambda_1}$ serve as linearly independent pair of functions. Similarly, g_{q,λ_1} and $\frac{\partial g_{q,\lambda_1}}{\partial \lambda_1}$ serve as linearly independent pair of functions.
  Now suppose for example that λ_1 appears with multiplicity k, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_k$. Then the functions $f_{q,\lambda_1}, \dots, f_{q,\lambda_k}$ and $g_{q,\lambda_1}, \dots, g_{q,\lambda_k}$ have to be substituted by the expressions $f_{q,\lambda_1}, \frac{\partial f_{q,\lambda_1}}{\partial \lambda_1}, \dots, \frac{\partial^{k-1} f_{q,\lambda_1}}{\partial \lambda_1^k}$ and $g_{q,\lambda_1}, \frac{\partial g_{q,\lambda_1}}{\partial \lambda_1}, \dots, \frac{\partial^{k-1} g_{q,\lambda_i}}{\partial \lambda_1^k}$ in Theorem 2. Similarly, Theorem 2 is adapted if other values λ_i appear with multiplicity k_i. Notice that λ_i are a priori fixed values. The derivative is understood in the symbolic sense (as if λ_i was a variable).
- Finally, suppose that at least one of the zeroes of the polynomial $P(\mathbf{D})$ is equal to zero. Assume without loss of generality that $\lambda_1 = 0$. Considering the limit behavior $\lambda_1 \to 0$ in Lemma 2, then the Laurent series expression simplifies to the usual one for monogenic functions [13]. This has the form

  $$f(\mathbf{z}) = \sum_{q=0}^{+\infty} P_q(\mathbf{z}) + \sum_{q'=0}^{+\infty} \frac{\mathbf{z}}{\|\mathbf{z}\|^{n+2q}} P'_{q'}(\mathbf{z}).$$

  For the case where one single value λ_i is equal to zero one replaces in Theorem 1 and in Theorem 2 the expressions $f_{q,0}$ by 1 and $g_{q,0}$ by $\frac{\mathbf{z}}{\|\mathbf{z}\|^{n+2q}}$.
  Further suppose that $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$. In this case the expressions f_{q,λ_i} $(i = 1, \dots, k)$ can be replaced by the expressions $1, \mathbf{z}, \mathbf{z}^2, \dots, \mathbf{z}^{k-1}$. This follows directly from applying the well-known Almansi–Fischer decomposition for

Clifford algebra-valued functions that are in Ker $\mathbf{D}^k$, see for instance [13,21]. The latter one states that every function f in Ker $\mathbf{D}^k$ has the local representation of the form $f_1 + \mathbf{z} f_2 + \cdots + \mathbf{z}^{k-1} f_k$, where $f_1, \dots f_k$ are solutions to Ker $\mathbf{D}$. In the case where n is odd the expressions g_{q,λ_i} $(i = 1, \dots, k)$ can be substituted by $\frac{\mathbf{z}}{\|\mathbf{z}\|^{n+2q}}, \frac{\mathbf{z}^2}{\|\mathbf{z}\|^{n+2q}}, \dots, \frac{\mathbf{z}^k}{\|\mathbf{z}\|^{n+2q}}$. However, in the case where n is even, one has to distinguish between two cases concerning the replacement of the functions g_{q,λ_i}. If $k \leqslant n + 2q - 1$, then the expressions g_{q,λ_i} $(i = 1, \dots, k)$ can also be substituted by $\frac{\mathbf{z}}{\|\mathbf{z}\|^{n+2q}}, \frac{\mathbf{z}^2}{\|\mathbf{z}\|^{n+2q}}, \dots, \frac{\mathbf{z}^k}{\|\mathbf{z}\|^{n+2q}}$. In the case $k = n + 2q - 1$ we deal with the expression $\mathbf{z}^{-1}$. Following [24] p. 104 in the case $k = n + 2q$ we can put $g_{q,\lambda_k}(\mathbf{z}) = \ln(\|\mathbf{z}\|)$. This expression is up to a constant the fundamental solution to $\Delta^{(n/2)}$. Indeed, by a direct computation one obtains that $\mathbf{D_z} \ln(\|\mathbf{z}\|) = c\mathbf{z}^{-1}$ where c is a real constant.
Following further [24], p. 104, formula (II.2.11) in the cases $k = n + 2q + 2m$ ($m \in \mathbb{N}$, n even) one can replace the functions q_{q,λ_k} by the expressions $\|\mathbf{z}\|^{2m} \ln(\|\mathbf{z}\|)$. We have $\mathbf{D_z}[\|\mathbf{z}\|^{2m} \ln(\|\mathbf{z}\|)] = C\mathbf{z}^{2m-1} \ln(\|\mathbf{z}\|)$, where C is a real constant. Summarizing, for all $k \geqslant n + 2q$ we can replace the functions g_{q,λ_k} by $\mathbf{z}^{k-(n+2q)} \ln(\|\mathbf{z}\|)$.

Next we want to establish the analogous results for the Szegö kernel:

Theorem 3. *Let $0 < \mu < 1$, $\lambda \in \mathbb{C} \setminus \{0\}$ and suppose that $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ are mutually distinct. Then the Szegö kernel of the annulus with radii $r = \mu$ and $R = 1$ with respect to the operator $(\mathbf{D_z} - \lambda)$ is given by*

$$S_\lambda(\mathbf{z}, \mathbf{w}) = \sum_{q=0}^{\infty} (f_{q,\lambda}, g_{q,\lambda})(\mathbf{z}) S_q(\mathbf{z}, \mathbf{w}) \big[\mathcal{T}_\lambda^{-1}\big] \begin{pmatrix} \overline{f}_{q,\lambda^\sharp} \\ \overline{g}_{q,\lambda^\sharp} \end{pmatrix} (\mathbf{w}),$$

where $\mathcal{T}_\lambda = (t_{ij})_{i,j=1,2}$ is the matrix with the entries

$$\begin{aligned}
t_{11} &= Sc\big\{\overline{f}_{q,\lambda^\sharp}(\mathbf{w}) f_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\big\{\overline{f}_{q,\lambda^\sharp}(\mathbf{w}) f_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=\mu},\\
t_{21} &= Sc\big\{\overline{g}_{q,\lambda^\sharp}(\mathbf{w}) f_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\big\{\overline{g}_{q,\lambda^\sharp}(\mathbf{w}) f_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=\mu},\\
t_{12} &= Sc\big\{\overline{f}_{q,\lambda^\sharp}(\mathbf{w}) g_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\big\{\overline{f}_{q,\lambda^\sharp}(\mathbf{w}) g_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=\mu},\\
t_{22} &= Sc\big\{\overline{g}_{q,\lambda^\sharp}(\mathbf{w}) g_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\big\{\overline{g}_{q,\lambda^\sharp}(\mathbf{w}) g_{q,\lambda}(\mathbf{w})\big\}\big|_{\|\mathbf{w}\|=\mu}.
\end{aligned}$$

The Szegö kernel of this annulus for the operator $(\mathbf{D_z} - \lambda_1)(\mathbf{D_z} - \lambda_2)$ has the form

$$S_{\lambda_1,\lambda_2}(\mathbf{z}, \mathbf{w}) = \sum_{q=0}^{\infty} (f_{q,\lambda_1}, f_{q,\lambda_2}, g_{q,\lambda_1}, g_{q,\lambda_2})(\mathbf{z}) S_q(\mathbf{z}, \mathbf{w}) \big[\mathcal{T}_{\lambda_1,\lambda_2}^{-1}\big] \begin{pmatrix} \overline{f}_{q,\lambda_1^\sharp} \\ \overline{f}_{q,\lambda_2^\sharp} \\ \overline{g}_{q,\lambda_1^\sharp} \\ \overline{g}_{q,\lambda_2^\sharp} \end{pmatrix} (\mathbf{w}),$$

where $\mathcal{T}_{\lambda_1,\lambda_2} = (t_{ij})_{i,j=1,\dots,4}$ is the matrix with the entries

$$t_{ij} = \begin{cases}
Sc\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w}) f_{q,\lambda_j}(\mathbf{w})\}|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w}) f_{q,\lambda_j}(\mathbf{w})\}|_{\|\mathbf{w}\|=\mu}, & i, j \in \{1, 2\},\\
Sc\{\overline{g}_{q,\lambda_{i-2}^\sharp}(\mathbf{w}) f_{q,\lambda_j}(\mathbf{w})\}|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\{\overline{g}_{q,\lambda_{i-2}^\sharp}(\mathbf{w}) f_{q,\lambda_j}(\mathbf{w})\}|_{\|\mathbf{w}\|=\mu}, & i \in \{3, 4\},\ j \in \{1, 2\},\\
Sc\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w}) g_{q,\lambda_{j-2}}(\mathbf{w})\}|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w}) g_{q,\lambda_{j-2}}(\mathbf{w})\}|_{\|\mathbf{w}\|=\mu}, & i \in \{1, 2\},\ j \in \{3, 4\},\\
Sc\{\overline{g}_{q,\lambda_{i-2}^\sharp}(\mathbf{w}) g_{q,\lambda_{j-2}}(\mathbf{w})\}|_{\|\mathbf{w}\|=1} - \mu^{n-1} Sc\{\overline{g}_{q,\lambda_{i-2}^\sharp}(\mathbf{w}) g_{q,\lambda_{j-2}}(\mathbf{w})\}|_{\|\mathbf{w}\|=\mu}, & i, j \in \{3, 4\}
\end{cases}$$

$$= \begin{cases}
J_{q+\frac{n}{2}-1}(\lambda_i^\sharp) J_{q+\frac{n}{2}-1}(\lambda_j) + J_{q+\frac{n}{2}}(\lambda_i^\sharp) J_{q+\frac{n}{2}}(\lambda_j)\\
\quad - \mu^{n-1}\big(J_{q+\frac{n}{2}-1}(\mu\lambda_i^\sharp) J_{q+\frac{n}{2}-1}(\mu\lambda_j) + J_{q+\frac{n}{2}}(\mu\lambda_i^\sharp) J_{q+\frac{n}{2}}(\mu\lambda_j)\big),\\
Y_{q+\frac{n}{2}-1}(\lambda_i^\sharp) J_{q+\frac{n}{2}-1}(\lambda_j) + Y_{q+\frac{n}{2}}(\lambda_i^\sharp) J_{q+\frac{n}{2}}(\lambda_j)\\
\quad - \mu^{n-1}\big(Y_{q+\frac{n}{2}-1}(\mu\lambda_i^\sharp) J_{q+\frac{n}{2}-1}(\mu\lambda_j) + Y_{q+\frac{n}{2}}(\mu\lambda_i^\sharp) J_{q+\frac{n}{2}}(\mu\lambda_j)\big),\\
J_{q+\frac{n}{2}-1}(\lambda_i^\sharp) Y_{q+\frac{n}{2}-1}(\lambda_j) + J_{q+\frac{n}{2}}(\lambda_i^\sharp) Y_{q+\frac{n}{2}}(\lambda_j)\\
\quad - \mu^{n-1}\big(J_{q+\frac{n}{2}-1}(\mu\lambda_i^\sharp) Y_{q+\frac{n}{2}-1}(\mu\lambda_j) + J_{q+\frac{n}{2}}(\mu\lambda_i^\sharp) Y_{q+\frac{n}{2}}(\mu\lambda_j)\big),\\
Y_{q+\frac{n}{2}-1}(\lambda_i^\sharp) Y_{q+\frac{n}{2}-1}(\lambda_j) + Y_{q+\frac{n}{2}}(\lambda_i^\sharp) Y_{q+\frac{n}{2}}(\lambda_j)\\
\quad - \mu^{n-1}\big(Y_{q+\frac{n}{2}-1}(\mu\lambda_i^\sharp) Y_{q+\frac{n}{2}-1}(\mu\lambda_j) + Y_{q+\frac{n}{2}}(\mu\lambda_i^\sharp) Y_{q+\frac{n}{2}}(\mu\lambda_j)\big).
\end{cases}$$

Proof. It suffices to treat the case $p = 2$ in detail, since the proof for the formula for the case $p = 2$ can directly be adapted to the case $p = 1$. The whole proof can be performed in analogy to the case of the Bergman kernel. Instead of the volume integrals we instead have to consider surface integrals over the expressions

$$S_q(\mathbf{z},\mathbf{w})\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v}),$$
$$S_q(\mathbf{z},\mathbf{w})\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v}),$$
$$S_q(\mathbf{z},\mathbf{w})\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v}),$$
$$S_q(\mathbf{z},\mathbf{w})\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})$$

extended over the two spheres bounding the annulus. Similarly to the proof of Theorem 2 we obtain that

$$\int\limits_{\mathbf{w}\in\partial B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w})\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\,dS_{\mathbf{w}} = \delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1},$$
$$\int\limits_{\mathbf{w}\in\partial B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w})\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\,dS_{\mathbf{w}} = \delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1},$$
$$\int\limits_{\mathbf{w}\in\partial B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w})\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\,dS_{\mathbf{w}} = \delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1},$$
$$\int\limits_{\mathbf{w}\in\partial B(0,\mu,1)} S_q(\mathbf{z},\mathbf{w})\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})S_{q'}(\mathbf{w},\mathbf{v})\,dS_{\mathbf{w}} = \delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1},$$

and a summation over $q = 0, 1, 2, 3, \ldots$ leads to

$$\sum_{q=0}^{\infty}\delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1} = \|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q,\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1} = S_q(\mathbf{z},\mathbf{v})t_{ij}.$$

Furthermore,

$$\sum_{q=0}^{\infty}\delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})f_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1} = S_q(\mathbf{z},\mathbf{v})t_{i+2,j},$$
$$\sum_{q=0}^{\infty}\delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{f}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1} = S_q(\mathbf{z},\mathbf{v})t_{i,j+2},$$
$$\sum_{q=0}^{\infty}\delta_{qq'}\|\mathbf{w}\|^{n-1}S_q(\mathbf{z},\mathbf{v})Sc\big\{\overline{g}_{q,\lambda_i^\sharp}(\mathbf{w})g_{q',\lambda_j}(\mathbf{w})\big\}\Big|_{\|\mathbf{w}\|=\mu}^{1} = S_q(\mathbf{z},\mathbf{v})t_{i+2,j+2}.$$

The invertibility of the matrix $\mathcal{T}_{\lambda_1,\lambda_2}$ can be shown by applying the same arguments as in the proof of Theorem 2. One just replaces the volume integrals by the corresponding surface integrals. □

Remark. In the case where $\lambda_2 = \lambda_1$ one again replaces in Theorem 3 f_{q,λ_2} and g_{q,λ_2} by $\frac{\partial f_{q,\lambda_1}}{\partial\lambda_1}$ and $\frac{\partial g_{q,\lambda_1}}{\partial\lambda_1}$, respectively. Similarly, if one or both values λ_i equal zero, the functions f_{q,λ_i} and g_{q,λ_i} are substituted in the same way as described in the second part of the remark after Theorem 2.

Theorem 4. *The Hardy space of functions that satisfy inside the annulus with radii $r = \mu \in (0,1)$ and $R = 1$ the equation $[\prod_{j=1}^{p}(\mathbf{D_z} - \lambda_j)]f = 0$ and that have boundary values in L^2, has no reproducing Szegö kernel function for all $p > 2$.*

Proof. Let $p > 2$. To show the non-existence of the Szegö kernel for these cases we consider the following set of $2p$ functions:

$$f_{0,\lambda_j}(\mathbf{w}) = J_{\frac{n}{2}-1}(\lambda_j r) + \frac{\mathbf{w}}{\|\mathbf{w}\|}J_{\frac{n}{2}}(\lambda_j r), \quad j \in \{1,\ldots,p\},$$
$$g_{0,\lambda_j}(\mathbf{w}) = Y_{\frac{n}{2}-1}(\lambda_j r) + \frac{\mathbf{w}}{\|\mathbf{w}\|}Y_{\frac{n}{2}}(\lambda_j r), \quad j \in \{1,\ldots,p\},$$

where we set $r := \|\mathbf{w}\|$. The set of functions $f_{0,\lambda_j}, g_{0,\lambda_j}$ is linearly independent on the annulus. However, if we restrict these functions to the sphere $R = 1$ and to the sphere $r = \mu$, they turn out to be non-trivial linear combinations of the functions 1 and $\mathbf{w}$. The vector space that is generated by the functions 1 and $\mathbf{w}$ on the sphere $R = 1$ and by 1 and $\mathbf{w}$ on the sphere $r=\mu$, is four-dimensional. In the case where $p \geqslant 3$ the set of functions $f_{0,\lambda_j}, g_{0,\lambda_j}$ however consists at least of six elements and is linearly dependent. Therefore, there exists an $\alpha = (\alpha_1, \ldots, \alpha_{2p})^t \in \mathbb{C}^{2p} \setminus \{0\}$, such that

$$\left(\sum_{j=1}^{p} \alpha_j f_{0,\lambda_j}(\mathbf{w}) + \sum_{j=p+1}^{2p} \alpha_j g_{0,\lambda_{j-p}}(\mathbf{w})\right)\Bigg|_{\|\mathbf{w}\|=1,\mu} = 0.$$

The function

$$h := \sum_{j=1}^{p} \alpha_j f_{0,\lambda_j} + \sum_{j=p+1}^{2p} \alpha_j g_{0,\lambda_{j-p}}$$

satisfies $[\prod_{j=1}^{p}(\mathbf{D_z} - \lambda_j)]h = 0$, because by construction the functions $f_{0,\lambda_j}, g_{0,\lambda_j}$ lie in the kernel of $\prod_{j=1}^{p}(\mathbf{D_z} - \lambda_j)$.

Now suppose that there exists a reproducing Szegö kernel S for the associated Hardy space of the annulus. Since the function h is an element of that Hardy space, it follows that

$$h(\mathbf{z}) = \int\limits_{\mathbf{w} \in \partial B(0,\mu,1)} S(\mathbf{z},\mathbf{w}) h(\mathbf{w})\, dS_{\mathbf{w}}.$$

However, $h|_{\|\mathbf{w}\| \in \{1,\mu\}} \equiv 0$. As a consequence, $h \equiv 0$ on the complete annulus. This is a contradiction, because for mutually distinct values λ_j the set of functions $f_{0,\lambda_j}, g_{0,\lambda_j}$ is linearly independent in the inside of the annulus. This however means that $h \not\equiv 0$. $\square$

4. A concrete application to Helmholtz type equations

Let us consider the case $p=2$ and $\lambda_1 = -\lambda_2$. Applying the Clifford algebra calculus, the equation $(\mathbf{D}+\lambda_1)(\mathbf{D}-\lambda_1)u(\mathbf{z}) = -f(\mathbf{z})$ can be rewritten in the form $(\Delta + \lambda_1^2)u(\mathbf{z}) = f(\mathbf{z})$. If λ_1 is real, then we deal with the Helmholtz equation treated explicitly for the three-dimensional case in [16, p. 81]. The positive square root of λ_1 then has the physical interpretation as the wave number k. The solutions to the Helmholtz equation include the solutions to the time-harmonic Maxwell equations equation, see for instance [17,18,20]. If λ_1 is purely imaginary, say $\lambda_1 = i\Lambda_1$, then we deal with the Klein–Gordon equation in the time-independent case. In this case we can identify $\Lambda_1 = \frac{mc}{\hbar}$, where m stands for the mass, c for the speed of light and $\hbar$ for the Planck number, cf. [19].

Let us now consider the following concrete boundary value problem involving general complex values for λ_1. Suppose concretely that Ω is the annulus of radii $r = \mu \in]0,1[$ and $R=1$ in $\mathbb{R}^n$. Let f be a given function from the Sobolev space $W^{2,k}(\Omega)$, i.e. its kth derivative in the sense of Sobolev is square integrable over the annulus. Furthermore, suppose that g is a given function on the boundary of the annulus belonging to $W^{2,k+3/2}(\partial\Omega)$. As shown in [16, p. 81] concretely for the three-dimensional and for real λ_1, also for general $n \in \mathbb{N}$ and arbitrary complex numbers λ the solutions to the boundary value problem

$$\begin{aligned} &(\Delta + \lambda_1^2)u(\mathbf{z}) = f(\mathbf{z}) \quad \text{on } \Omega, \\ &u(\mathbf{z}) = g(\mathbf{z}) \quad \text{at } \partial\Omega \end{aligned}$$

can be expressed in terms of hypercomplex integral operators. Adapting the calculations from [16, pp. 81–83] to the slightly more general framework of considering complex values for λ_1 and general $n \in \mathbb{N}$, the solutions to the posed boundary value problem can still be written as

$$u = F_{\lambda_1} g + T_{-\lambda_1}\mathbf{P}_{\lambda_1}(\mathbf{D}-\lambda_1)h - T_{-\lambda_1}(I - \mathbf{P}_{\lambda_1})T_{\lambda_1} f \in W^{2,k+2}(\Omega), \tag{4}$$

where h is an $W^{2,k+2}(\Omega)$-extension of g. We have a unique solution in all cases where $\Im(\lambda_1) \neq 0$. If λ_1 is real then we have uniqueness if λ_1^2 is not an eigenvalue of $-\Delta$. In the representation (4) the letter I stands for the identity operator, T_{λ_1} is the Teodorescu transform, F_{λ_1} the Cauchy transform and $\mathbf{P}_{\lambda_1}$ is the Bergman projection for the operator $\mathbf{D} - \lambda_1$ associated to the domain Ω. The Teodorescu transform and the Cauchy transform have independently of the domain Ω in the general n-dimensional case the following universal representation

$$(T_\lambda u)(\mathbf{z}) = -\int\limits_{\Omega} e_\lambda(\mathbf{z}-\mathbf{w}) u(\mathbf{w})\, dV_{\mathbf{w}}, \quad \mathbf{z} \in \mathbb{R}^n,$$

and

$$(F_\lambda u)(\mathbf{z}) = \int\limits_{\partial\Omega} e_\lambda(\mathbf{z}-\mathbf{w}) n(\mathbf{w}) u(\mathbf{w})\, dS_{\mathbf{w}}, \quad \mathbf{z} \in \mathbb{R}^n \setminus \partial\Omega,$$

respectively. Here,

$$e_\lambda(\mathbf{z}) = \begin{cases} \frac{\pi i}{A_n \Gamma(n/2)} (\frac{\lambda}{2})^{n/2} \|\mathbf{z}\|^{1-n/2} \big[H^{(1)}_{n/2-1}(\lambda\|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} H^{(1)}_{n/2}(\lambda\|\mathbf{z}\|)\big], & \Im(\lambda) > 0, \\ \frac{-\pi i}{A_n \Gamma(n/2)} (\frac{\lambda}{2})^{n/2} \|\mathbf{z}\|^{1-n/2} \big[H^{(2)}_{n/2-1}(\lambda\|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} H^{(2)}_{n/2}(\lambda\|\mathbf{z}\|)\big], & \Im(\lambda) < 0, \\ \frac{\pi}{A_n \Gamma(n/2)} (\frac{\lambda}{2})^{n/2} \|\mathbf{z}\|^{1-n/2} \big[Y_{n/2-1}(\lambda\|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} Y_{n/2}(\lambda\|\mathbf{z}\|)\big], & \Im(\lambda) = 0 \end{cases}$$

is the fundamental solution to $(\mathbf{D} - \lambda)u = 0$ in $\mathbb{R}^n$, cf. [27]. The functions $H^{(1)}$ and $H^{(2)}$ stand for the Hankel functions, defined by

$$H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z), \quad \nu \in \frac{1}{2}\mathbb{N}, \ z \in \mathbb{C},$$
$$H^{(2)}_\nu(z) = J_\nu(z) - iY_\nu(z),$$

see [15] for details. In the case $\lambda = 0$ the expression $e_\lambda(\mathbf{z})$ reduces to the usual Cauchy kernel function, i.e. $-\frac{\mathbf{z}}{\|\mathbf{z}\|^n}$ ($n = 3$ in the three-dimensional case).

The Bergman projection $\mathbf{P}$ however depends on the domain. In the case where Ω is the annulus of radii $r = \mu \in]0, 1[$ and $R = 1$, it has the concrete form

$$[\mathbf{P}_{\lambda_i} h](\mathbf{z}) = \int_\Omega B_{\lambda_i}(\mathbf{z}, \mathbf{w}) h(\mathbf{w})\, dV_{\mathbf{w}}, \quad i = 1, \ldots, p,$$

where $B_{\lambda_i}(\mathbf{z}, \mathbf{w})$ $(i = 1, \ldots, p)$ is precisely the Bergman kernel function of the annulus for the operator $(\mathbf{D} - \lambda_i)$ that we computed in the previous section. In the particular case $\lambda_i = 0$ the Bergman kernel can be obtained by substituting in $f_{q,0} =: 1$ and $g_{q,0} := \frac{\mathbf{z}}{\|\mathbf{z}\|^{n+2q}}$ into Theorem 1. Alternatively, we can use for the Bergman kernel of the annulus associated to Ker $\mathbf{D}$ the series representation formula

$$B(\mathbf{z}, \mathbf{w})_0 := \frac{1}{(n-2)A_n} \sum_{k \in \mathbb{Z}} D_z \frac{\mu^{k(n-2)}}{\|1 + \mu^{2k} \mathbf{z}\mathbf{w}\|^{n-2}} D_w,$$

that we determined in our previous paper [8]. The explicit knowledge of the Bergman kernel of the annulus for elements in Ker $(\mathbf{D} - \lambda)$ thus enables us to evaluate the representation formula (4) fully explicitly and allows us to compute the solutions to the posed boundary value problem in an analytic way. In our follow-up paper [9] we treat more generally Dirichlet problems of the form $P(\mathbf{D})u = f$ where $P(\mathbf{D})$ is an arbitrary polynomial in $\mathbf{D}$ with complex coefficients. These include the boundary value problems treated here as special cases. The treatment of this more general class of boundary value problems however requires more sophisticated techniques than the Helmholtz type equations considered here. Therefore, this topic will be treated in a separate paper. In this more general context we need to apply the formulas for the Bergman kernel for polynomial Dirac equations of general degree which are developed in Theorem 2.

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